Renormalization group improvement of the effective potential in massive ϕ^4 theory: next-next-next-to-leading logarithm resummation

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Abstract

The renormalization group method is applied to the three-loop effective potential of the massive ϕ^4 theory in the $\overline{\mathrm{MS}}$ scheme in order to obtain the next-next-next-to leading logarithm resummation. For this, we use already known four-loop renormalization group functions and calculate perturbatively evolutions of the parameters (λ, m^2, ϕ and, Λ) along the running scale within the accuracy of the three-loop order. We also comment on the structure of five-loop effective potential using the renormalization group equation for the effective potential and the existing five-loop renormalization group functions.

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I. INTRODUCTION

The effective potential [1–3] plays a crucial role in probing the vacuum structure of quantum field theory. The usual way of computing the effective potential is a loop expansion, for which an elegant method called the field shift method was developed by Jackiw [2]. This method enables us to avoid an onerous task of summing, at each loop order, infinite series of Feynman diagrams for which the combinatorial factors are very complicated, especially when several interactions are present. The calculations of the effective potential for a single-component massive ϕ^4 theory [4] and for a massless $O(N)$ ϕ^4 theory [5,6] were achieved at the three-loop level in four dimensions of spacetime.

The renormalization group (RG) method has proved one of the most important tools in refined perturbative analyses. The concept of the GR improved perturbation theory was originally introduced long ago within the context of QED in the landmark work of Gell-Mann and Low [7]. In the expression of an RG improved quantity, whether it be the Green's function, effective potential, or any other quantity predictable from Feynman diagram perturbation theory, the bare parameters in the corresponding expression are replaced with their scale-dependent running forms usually calculated to some given order in the perturbation theory.

In one of the early applications of the RG method, Coleman and Weinberg [1] considered the effective potential $V(\phi)$ for a spacetime-independent scalar field ϕ in the context of massless models. In the massive case it has been demonstrated that this treatment also works provided one takes into account a nontrivial running of vacuum energy [8–11]. Whilst in flat spacetime this running of vacuum energy is more a tool of calculational convenience, in curved spacetime it describes the running of the cosmological constant [12].

In this paper, we extend earlier work [13], in which the next-next-to-leading logarithm series expansion of the effective potential for a single-component massive ϕ^4 theory was obtained, to the next-next-next-to-leading logarithm order. In Sec. II, the analytical evaluation of *finite* parts of three-loop diagrams is explained in detail. In particular, these finite parts of the three-loop diagrams are expressed in terms of known transcendental numbers [16–19]. The finite parts, as well as pole parts, of highest-loop diagrams in a given order calculation, are needed in a mass-independent renormalization scheme such as a minimal subtraction (MS) [14] or a modified minimal subtraction ($\overline{\text{MS}}$) [15] scheme. In the latter part of this section, without discussing the technical details of how the effective potential is computed, a summary of the $\overline{\text{MS}}$ three-loop effective potential for a single-component massive ϕ^4 theory is given. In Sec. III, the perturbation solutions for running parameters $\bar{\lambda}(t)$, $\bar{m}^2(t)$, $\bar{\phi}(t)$, and $\bar{\Lambda}(t)$ are obtained and the result of next-next-next-to-leading logarithm series is reported. The final section is devoted to the concluding remarks.

II. THREE-LOOP EFFECTIVE POTENTIAL IN THE $\overline{\text{MS}}$ SCHEME

The Lagrangian of the massive ϕ^4 theory is given as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 - \Lambda \\ &+\frac{\delta Z}{2}(\partial\phi)^2 - \frac{\delta m^2}{2}\phi^2 - \frac{\delta\lambda}{4!}\phi^4 - \delta\Lambda\ . \end{aligned} \tag{1}$$

Here δZ, δm^2, and $\delta\lambda$ are the so-called counterterms of the wave function, mass, and coupling constant, respectively. Three-loop effective potential of this theory was calculated [4] in the framework of the dimensional regularization [20], in which an arbitrary constant, μ, with mass dimension is introduced inevitably for a dimensional reason. The subtraction done in Ref. [4] is nonminimal. This means that various counterterms of each loop order contain mass-dependent arbitrary finite terms as well as ϵ-pole terms. These finite parts of counterterms are determined by imposing the renormalization conditions on the effective potential at a given renormalization scale.[1] Therefore, in the mass-dependent scheme we do not need to calculate finite parts of three-loop diagrams. Knowledge of pole terms is sufficient.

However, in a mass-independent scheme such as the MS or $\overline{\text{MS}}$ scheme, we have to calculate three-loop diagrams to the ϵ^0 order. Without imposing renormalization conditions at a specfic scale, we just leave μ unspecified, as in Eq. (12) below. This has the drawback that it does not involve true physical parameters measured at a given scale. Though it normally takes some effort to express physically measurable quantities in terms of the parameters of the $\overline{\text{MS}}$ expression, the RG equation is dealt with much easier and the calculations in complicated theories are much more convenient.

A. Analytic evaluation of finite parts of three-loop diagrams

Let us define three-loop vacuum integrals J, K, L, and M, which are nonfactorizable into lower-loop integrals:[2]

$$J \equiv \int_{kpq} \frac{1}{(p^2+m_\phi^2)[(p+k)^2+m_\phi^2](q^2+m_\phi^2)[(q+k)^2+m_\phi^2]} \ ,$$

---

[1] We stress that the dimensional regularization is perfectly possible *with* renormalization conditions; the renormalized quantities such as effective potentials or Green's functions would then be identically the same as those found by regularization with a cutoff: they depend only on the renormalization conditions, and not on the regularization procedure. The calculations of effective potential in Ref. [4] and Ref. [5] are done in the dimensional regularization scheme, with a specific set of renormalization conditions. The same calculations at lower-loop level, in the cutoff regularization, with the same renormalization conditions can be found in Ref. [3] and Ref. [2] respectively. We see that the results agree with each other.

[2] There are two factorizable three-loop vacuum integrals to be calculated. The exact analytical values of the component one-loop and two-loop integrals are known. We mention briefly only two-loop integral

$$\int_{pq} \frac{1}{(p^2+m_\phi^2)^2(q^2+m_\phi^2)[(p+q)^2+m_\phi^2]} = -\frac{1}{3}\frac{\partial}{\partial m_\phi^2}\int_{pq} \frac{1}{(p^2+m_\phi^2)(q^2+m_\phi^2)[(p+q)^2+m_\phi^2]} \ .$$

The exact values of $\int_{pq} \frac{1}{(p^2+m_\phi^2)(q^2+m_\phi^2)[(p+q)^2+m_\phi^2]}$ can be found in a recent paper by Davydychev [19]. See also Ref. [21] and Eq. (7) below.

$$K \equiv \int_{kpq} \frac{1}{(k^2+m_\phi^2)(p^2+m_\phi^2)[(p+k)^2+m_\phi^2](q^2+m_\phi^2)[(q+k)^2+m_\phi^2]} \;,$$
$$L \equiv \int_{kpq} \frac{1}{(k^2+m_\phi^2)^2(p^2+m_\phi^2)[(p+k)^2+m_\phi^2](q^2+m_\phi^2)[(q+k)^2+m_\phi^2]} \;,$$
$$M \equiv \int_{kpq} \frac{1}{(k^2+m_\phi^2)(p^2+m_\phi^2)(q^2+m_\phi^2)[(k-p)^2+m_\phi^2][(p-q)^2+m_\phi^2][(q-k)^2+m_\phi^2]} \;. \quad (2)$$

The momenta in Eq. (2) are all (Wick-rotated) Euclidean ones and the abbreviated integration measure is defined as

$$\int_k = \mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \;, \quad (3)$$

where $n = 4-\epsilon$ is the space-time dimension in the frame work of dimensional regularization, and μ is an arbitrary constant with mass dimension as mentioned above. Though m_ϕ^2 in Eq. (2) is defined as the constant-field-dependent mass squared parameter

$$m_\phi^2 \equiv m^2 + \frac{\lambda\phi^2}{2} \;,$$

it can be viewed as a mere positive constant in this subsection. The pole parts of the integrals in Eq. (2) are known [4,6]. We now calculate finite parts of these integrals in terms of known transcendental numbers. (In fact, the analytical expression of the finite part of M is a simple quotation from Ref. [16].)

Whilst massless multi-loop diagrams can be dealt with by essentially algebraic methods [22], the situation is more complicated in the case of massive diagrams. In notation of Avdeev [23], the above three-loop diagrams J, K, and L correspond to $B_N(0,0,1,1,1,1)$, $D_5(1,1,1,1,1,0)$, and $D_5(1,1,1,1,2,0)$, respectively. The recurrence relations, given in Avdeev's paper [23], allows us to connect the integrals $B_N(0,0,1,1,1,1)$, $D_5(1,1,1,1,1,0)$, and $D_5(1,1,1,1,2,0)$ with the tetrahedron integrals $B_N(1,1,1,1,1,1)$, $B_M(1,1,1,1,1,1)$, and $D_5(1,1,1,1,1,1)$. The analytic expressions of the finite parts, i.e., ϵ^0 order terms, for all tetrahedron vacuum diagrams with different combinations of massless and massive lines of a single mass scale are known in a recent paper by Broadhurst [16]. With a convention of integration measure used in Ref. [16]:

$$\int [dk] = \int \frac{d^n k}{m_\phi^{n-4} \pi^{n/2} \Gamma(1+\varepsilon)},$$

where $n = 4-2\varepsilon$, we quote the results of Broadhurst which are relevant to our calculation:

$$\begin{aligned}
B_N(1,1,1,1,1,1) &= V_{4N} = \frac{2\zeta(3)}{\varepsilon} + 6\zeta(3) - 14\zeta(4) - 16U_{3,1} \;, \\
B_M(1,1,1,1,1,1) &= V_{3T} = \frac{2\zeta(3)}{\varepsilon} + 6\zeta(3) - 9\zeta(4) \;, \\
D_5(1,1,1,1,1,1) &= V_5 = \frac{2\zeta(3)}{\varepsilon} + 6\zeta(3) - \frac{469}{27} + \frac{8}{3}\mathrm{Cl}_2^2(\pi/3) - 16V_{3,1} \;, \\
D_6(1,1,1,1,1,1) &= V_6 = \frac{2\zeta(3)}{\varepsilon} + 6\zeta(3) - 13\zeta(4) - 8U_{3,1} - 4\mathrm{Cl}_2^2(\pi/3) \;,
\end{aligned} \quad (4)$$

where $U_{3,1}$ and $V_{3,1}$ are defined as

$$U_{3,1} \equiv \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n} \, ,$$
$$V_{3,1} \equiv \sum_{m>n>0} \frac{(-1)^m \cos(2\pi n/3)}{m^3 n} \, ,$$

and can be expressed in terms of known transcendental numbers:[3]

$$\begin{aligned} U_{3,1} &= \frac{\zeta(4)}{2} + \frac{\zeta(2)}{2}\ln^2 2 - \frac{1}{12}\ln^4 2 - 2\mathrm{Li}_4\left(\frac{1}{2}\right) \, , \\ V_{3,1} &= \frac{1}{3}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right) - \frac{1}{4}\pi \mathrm{Ls}_3\left(\frac{2\pi}{3}\right) + \frac{13}{24}\zeta(3)\ln 3 - \frac{259}{108}\zeta(4) + \frac{3}{8}\mathrm{Ls}_4^{(1)}\left(\frac{2\pi}{3}\right) \, . \end{aligned} \tag{5}$$

In the above equation, $\mathrm{Li}_4(x)$, $\mathrm{Cl}_2(x)$, $\mathrm{Ls}_3(x)$, and $\mathrm{Ls}_4^{(1)}(x)$ are the polylogarithm, Clausen's polylogarithm, log-sine integral, and generalized log-sine integral respectively [24], whose numerical values at the given arguments are

$$\begin{aligned} \mathrm{Li}_4\left(\frac{1}{2}\right) &= 5.972\,734\,619\,850\,133... \, , \\ \mathrm{Cl}_2\left(\frac{\pi}{3}\right) &= 1.014\,941\,606\,409\,653... \, , \\ \mathrm{Ls}_3\left(\frac{2\pi}{3}\right) &= -2.144\,767\,212\,569\,494... \, , \\ \mathrm{Ls}_4^{(1)}\left(\frac{2\pi}{3}\right) &= -0.497\,675\,551\,606\,647... \, . \end{aligned}$$

Using the the method of recurrence relations [22,23,25] ingeniously enough, we can arrive at following connections:

$$\begin{aligned} m_\phi^{-4} B_N(0,0,1,1,1,1) &= 32 B_4\left(\frac{1}{2(1-3\varepsilon)} + \frac{4}{2-3\varepsilon} - \frac{2}{1-2\varepsilon} - \frac{1}{2(1-\varepsilon)}\right) \\ &- \frac{486}{1-3\varepsilon} - \frac{729}{2(2-3\varepsilon)} - \frac{35}{2\varepsilon} + \frac{3}{\varepsilon^2} + \frac{2}{\varepsilon^3} + \frac{512}{1-2\varepsilon} + \frac{10}{1-\varepsilon} \\ &+ \frac{\Gamma(1-\varepsilon)\Gamma^2(1+2\varepsilon)\Gamma(1+3\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(1+4\varepsilon)}\left(\frac{14}{3\varepsilon^2} + \frac{35}{\varepsilon} + \frac{378}{1-3\varepsilon} + \frac{189}{2-3\varepsilon} - \frac{896}{3(1-2\varepsilon)} - \frac{14}{3(1-\varepsilon)}\right) \, , \\ D_5(1,1,1,1,1,1) &= -\frac{2}{3} m^{-2} D_5(1,1,1,1,1,0)\left(1 - \frac{1}{2\varepsilon}\right) \\ &- \frac{8}{3} B_4\left(\frac{3}{2(1-3\varepsilon)} - \frac{2}{1-2\varepsilon} + \frac{1}{2(1-\varepsilon)}\right) - \frac{2}{3\varepsilon^2} m_\phi^{-2} \mathbf{VL111}(1,1,1) \\ &- \frac{2}{3\varepsilon^4} - \frac{1}{\varepsilon^3} + \frac{4}{3\varepsilon^2} + \frac{243}{2(1-3\varepsilon)} + \frac{15}{\varepsilon} - \frac{160}{3(1-2\varepsilon)} + \frac{7}{6(1-\varepsilon)} \end{aligned}$$

---

[3]The combination $V_{3,1}$ in Eq. (5) in terms of the known transcendental numbers was found by Fleischer and Kalmykov [18].

$$
-\frac{\Gamma(1-\varepsilon)\Gamma^2(1+2\varepsilon)\Gamma(1+3\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(1+4\varepsilon)}\left(\frac{7}{9\varepsilon^3}+\frac{14}{3\varepsilon^2}+\frac{175}{9\varepsilon}+\frac{189}{2(1-3\varepsilon)}-\frac{224}{9(1-2\varepsilon)}+\frac{7}{18(1-\varepsilon)}\right),
$$

$$
\begin{aligned}
D_5(1,1,1,1,2,0) &= \frac{1}{3}m_\phi^{-2}D_5(1,1,1,1,1,0)(1+\varepsilon)-\frac{32}{3}B_4\left(\frac{1}{2(1-3\varepsilon)}-\frac{1}{1-2\varepsilon}+\frac{1}{2(1-\varepsilon)}\right) \\
&-\frac{4}{3\varepsilon}m_\phi^{-2}\mathbf{VL111}(1,1,1)+\frac{162}{1-3\varepsilon}+\frac{44}{3\varepsilon}-\frac{4}{3\varepsilon^3}-\frac{256}{3(1-2\varepsilon)}+\frac{10}{3(1-\varepsilon)} \\
&-\frac{\Gamma(1-\varepsilon)\Gamma^2(1+2\varepsilon)\Gamma(1+3\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(1+4\varepsilon)}\left(\frac{28}{9\varepsilon^2}+\frac{56}{3\varepsilon}+\frac{126}{1-3\varepsilon}-\frac{448}{9(1-2\varepsilon)}+\frac{14}{9(1-\varepsilon)}\right), \qquad (6)
\end{aligned}
$$

where B_4 is proportional to the difference between $B_N(1,1,1,1,1,1)$ and $B_M(1,1,1,1,1,1,)$ [25]:

$$
B_4=-\frac{(1-2\varepsilon)(2-2\varepsilon)}{4}\Big(B_M(1,1,1,1,1,1)-B_N(1,1,1,1,1,1)\Big)
$$

and it is reduced to a ${}_3F_2$ hypergeometric series:

$$
\begin{aligned}
B_4 &= \frac{7}{24\varepsilon^4}\left(1-\frac{\Gamma(1-\varepsilon)\Gamma^2(1+2\varepsilon)\Gamma(1+3\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(1+4\varepsilon)}\right)-\frac{\pi^2}{3\varepsilon^2}\frac{\Gamma(1+2\varepsilon)\Gamma(1+3\varepsilon)}{2^{6\varepsilon}\Gamma^5(1+\varepsilon)} \\
&+\frac{8}{3\varepsilon^2(1+2\varepsilon)}\ {}_3F_2\left(1,\frac{1}{2}-\varepsilon,\frac{1}{2}-\varepsilon;\frac{3}{2}+\varepsilon,\frac{3}{2};1\right),
\end{aligned}
$$

whose lowest three orders in ε expansion can be found in Eq. (43) of Ref. [26]. It turns out that, since B_4 starts with an ε^0 order term in ε expansion, without pole terms, the value of B_4 does not play any role in our calculation pursued to the accuracy of ε^0 order. In Eq. (6), **VL111**(1,1,1) denotes a two-loop bubble integral, shown in Fig. 1 of the paper by Fleischer and Kalmykov [18]. Its value up to ε^3 order in ε expansion can be found in Eq. (7) of Ref. [18]. For all higher order terms, one may refer to Ref. [19]. It is sufficient to take its value up to ε order to our desired accuracy:

$$
\begin{aligned}
m_\phi^{-2}\mathbf{VL111}(1,1,1) &= -\frac{3}{2(1-\varepsilon)(1-2\varepsilon)}\bigg[\frac{1}{\varepsilon^2}+2A \\
&+2\varepsilon\left\{-A\ln 3-\frac{\pi^3}{6\sqrt{3}}-\sqrt{3}\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)\right\}+O(\varepsilon^2)\bigg], \qquad (7)
\end{aligned}
$$

where numerical value of the transcendental number A is given as

$$
A\equiv-\frac{2}{\sqrt{3}}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)=-1.171\,953\,619\,344\,729\ldots.
$$

From Eqs. (4) – (7), we eventually obtain

$$
\begin{aligned}
B_N(0,0,1,1,1,1) &= \frac{2}{\varepsilon^3}+\frac{23}{3\varepsilon^2}+\frac{35}{2\varepsilon}+\frac{275}{12}+O(\varepsilon), \\
D_5(1,1,1,1,1,0) &= -\frac{1}{\varepsilon^3}-\frac{17}{3\varepsilon^2}+\frac{1}{\varepsilon}\left(-\frac{67}{3}-6A\right)-\frac{229}{3}-30A \\
&+6A\ln 3+\frac{\pi^3}{\sqrt{3}}+6\zeta(3)+6\sqrt{3}\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)+O(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
D_5(1,1,1,1,2,0) &= \frac{1}{3\varepsilon^3} + \frac{2}{3\varepsilon^2} + \frac{1}{\varepsilon}\Big(\frac{2}{3} + 2A\Big) \\
&\quad -\frac{2}{3} - 2A\ln 3 + 2\zeta(3) - \frac{\pi^3}{3\sqrt{3}} - 2\sqrt{3}\mathrm{Ls}_3\Big(\frac{2\pi}{3}\Big) + O(\varepsilon)\ , \\
D_6(1,1,1,1,1,1) &= \frac{2\zeta(3)}{\varepsilon} - 3A^2 + \frac{2}{3}\ln^4 2 - \frac{2\pi^2}{3}\ln^2 2 \\
&\quad +6\zeta(3) - 17\zeta(4) + 16\mathrm{Li}_4\Big(\frac{1}{2}\Big) + O(\varepsilon)\ . \qquad (8)
\end{aligned}
$$

We readily recover the values of J, K, L, and M in our original integration measure, Eq. (3), by the following relations:

$$
\begin{aligned}
J &= \frac{m_\phi^4}{(4\pi)^6}\Big(\frac{m_\phi^2}{4\pi\mu^2}\Big)^{-3\epsilon/2} \exp\Big[-\frac{3\gamma_{\mathrm{E}}}{2}\epsilon + \frac{3\zeta(2)}{8}\epsilon^2 - \frac{\zeta(3)}{8}\epsilon^3 + \cdots\Big] B_N(0,0,1,1,1,1)\ , \\
K &= \frac{m_\phi^2}{(4\pi)^6}\Big(\frac{m_\phi^2}{4\pi\mu^2}\Big)^{-3\epsilon/2} \exp\Big[-\frac{3\gamma_{\mathrm{E}}}{2}\epsilon + \frac{3\zeta(2)}{8}\epsilon^2 - \frac{\zeta(3)}{8}\epsilon^3 + \cdots\Big] D_5(1,1,1,1,1,0)\ , \\
L &= \frac{1}{(4\pi)^6}\Big(\frac{m_\phi^2}{4\pi\mu^2}\Big)^{-3\epsilon/2} \exp\Big[-\frac{3\gamma_{\mathrm{E}}}{2}\epsilon + \frac{3\zeta(2)}{8}\epsilon^2 - \frac{\zeta(3)}{8}\epsilon^3 + \cdots\Big] D_5(1,1,1,1,2,0)\ , \\
M &= \frac{1}{(4\pi)^6}\Big(\frac{m_\phi^2}{4\pi\mu^2}\Big)^{-3\epsilon/2} \exp\Big[-\frac{3\gamma_{\mathrm{E}}}{2}\epsilon + \frac{3\zeta(2)}{8}\epsilon^2 - \frac{\zeta(3)}{8}\epsilon^3 + \cdots\Big] D_6(1,1,1,1,1,1)\ ,
\end{aligned}
$$

where the exp[ ] factor comes from an expansion of $\Gamma^3(1+\epsilon/2)$.

In the *standard* $\overline{\mathrm{MS}}$ scheme [15], the factors $\ln(4\pi)$ and γ_{E} are absorbed into the renormalization scale μ. But in other wide-spread convention of $\overline{\mathrm{MS}}$ (see, e.g., Refs. [27,28]), the factor $\zeta(2)$ is absorbed further into the scale μ. (This convention gives the same result in the one-loop diagrams and is more convenient in higher-loop massive calculations.) By introducing a new renormalization scale μ_*:

$$
\mu_*^2 = 4\pi\mu^2 \exp\Big(-\gamma_{\mathrm{E}} + \frac{\zeta(2)\epsilon}{4}\Big)\ , \qquad (9)
$$

the above four integrals J, K, L, and M are given as follows:

$$
\begin{aligned}
J &= \frac{m_\phi^4}{(4\pi)^6}\Big(\frac{m_\phi^2}{\mu_*^2}\Big)^{-3\epsilon/2}\Big[\frac{16}{\epsilon^3} + \frac{92}{3\epsilon^2} + \frac{35}{\epsilon} + F_J\Big]\ , \\
K &= \frac{m_\phi^2}{(4\pi)^6}\Big(\frac{m_\phi^2}{\mu_*^2}\Big)^{-3\epsilon/2}\Big[-\frac{8}{\epsilon^3} - \frac{68}{3\epsilon^2} + \frac{1}{\epsilon}\Big\{-\frac{134}{3} - 12A\Big\} + F_K\Big]\ , \\
L &= \frac{1}{(4\pi)^6}\Big(\frac{m_\phi^2}{\mu_*^2}\Big)^{-3\epsilon/2}\Big[\frac{8}{3\epsilon^3} + \frac{8}{3\epsilon^2} + \frac{1}{\epsilon}\Big\{\frac{4}{3} + 4A\Big\} + F_L\Big]\ , \\
M &= \frac{1}{(4\pi)^6}\Big(\frac{m_\phi^2}{\mu_*^2}\Big)^{-3\epsilon/2}\Big[\frac{4\zeta(3)}{\epsilon} + F_M\Big]\ , \qquad (10)
\end{aligned}
$$

where

$$
F_J = \frac{275}{12} - 2\zeta(3)\ ,
$$

$$
\begin{aligned}
F_K &= -\frac{229}{3} - 30A + 6A\ln 3 + \frac{\pi^3}{\sqrt{3}} + 7\zeta(3) + 6\sqrt{3}\mathrm{Ls}_3\Big(\frac{2\pi}{3}\Big)\,,\\
F_L &= -\frac{2}{3} - 2A\ln 3 - \frac{\pi^3}{3\sqrt{3}} + \frac{5\zeta(3)}{3} - 2\sqrt{3}\mathrm{Ls}_3\Big(\frac{2\pi}{3}\Big)\,,\\
F_M &= -3A^2 + 6\zeta(3) - 17\zeta(4) - \frac{2\pi^2}{3}\ln^2 2 + \frac{2}{3}\ln^4 2 + 16\mathrm{Li}_4\Big(\frac{1}{2}\Big)\,. \qquad (11)
\end{aligned}
$$

B. Summary of three-loop effective potential in $\overline{\mathrm{MS}}$ scheme

Once all values of diagrams needed for three-loop effective potential of the massive ϕ^4 theory are known, the calculation is straightforward albeit long. Thus we simply summarize the result:[4]

$$
\begin{aligned}
V &= V^{(0)} + \hbar V^{(1)} + \hbar^2 V^{(2)} + \hbar^3 V^{(3)} + O(\hbar^4)\,,\\
V^{(0)} &= \frac{m^2\phi^2}{2} + \frac{\lambda\phi^4}{4!} + \Lambda\,,\\
V^{(1)} &= \frac{\lambda}{(4\pi)^2}\Big[-\frac{3m^4}{8\lambda} - \frac{3m^2\phi^2}{8} - \frac{3\lambda\phi^4}{32} + \Big\{\frac{m^4}{4\lambda} + \frac{m^2\phi^2}{4} + \frac{\lambda\phi^4}{16}\Big\}\ln\Big(\frac{m_\phi^2}{\mu^2}\Big)\Big]\,,\\
V^{(2)} &= \frac{\lambda^2}{(4\pi)^4}\Big[\frac{m^4}{8\lambda} + m^2\phi^2\Big(\frac{3}{4} + \frac{A}{4}\Big) + \lambda\phi^4\Big(\frac{11}{32} + \frac{A}{8}\Big)\\
&\quad - \Big\{\frac{m^4}{4\lambda} + \frac{3m^2\phi^2}{4} + \frac{5\lambda\phi^4}{16}\Big\}\ln\Big(\frac{m_\phi^2}{\mu^2}\Big) + \Big\{\frac{m^4}{8\lambda} + \frac{m^2\phi^2}{4} + \frac{3\lambda\phi^4}{32}\Big\}\ln^2\Big(\frac{m_\phi^2}{\mu^2}\Big)\Big]\,,\\
V^{(3)} &= \frac{\lambda^3}{(4\pi)^6}\Big[Q_1\frac{m^4}{\lambda} + Q_2 m^2\phi^2 + Q_3\lambda\phi^4\\
&\quad + \Big\{\frac{41m^4}{96\lambda} + m^2\phi^2\Big(\frac{371}{96} + \frac{7A}{8}\Big) + \lambda\phi^4\Big(\frac{701}{384} + \frac{9A}{16} + \frac{\zeta(3)}{4}\Big)\Big\}\ln\Big(\frac{m_\phi^2}{\mu^2}\Big)\\
&\quad - \Big\{\frac{17m^4}{48\lambda} + \frac{37m^2\phi^2}{24} + \frac{143\lambda\phi^4}{192}\Big\}\ln^2\Big(\frac{m_\phi^2}{\mu^2}\Big)\\
&\quad + \Big\{\frac{5m^4}{48\lambda} + \frac{7m^2\phi^2}{24} + \frac{9\lambda\phi^4}{64}\Big\}\ln^3\Big(\frac{m_\phi^2}{\mu^2}\Big)\Big]\,, \qquad (12)
\end{aligned}
$$

where[5]

$$
Q_1 = \frac{7}{16} - \frac{F_J}{48} + \frac{\psi''(2)}{48} = 0.001\,736\cdots\,,
$$

---

[4]From now on, we omit the subscript $*$ in μ_*.

[5]The quoted numerical values of Q_1, Q_2, and Q_3 below Eq. (26) of Ref. [13] should be replaced with those in Eq. (13). This is because the previous values are calculated in the standard $\overline{\mathrm{MS}}$ scheme and the RG functions in Eq. (13) of Ref. [13] are obtained using the convention of Eq. (9).

$$
\begin{aligned}
Q_2 &= 6+\frac{17A}{8}-\frac{3A}{4}\ln 3-\frac{F_J}{48}+\frac{F_K}{8}\\
&\quad -\frac{\pi^3}{8\sqrt{3}}-\frac{\zeta(3)}{4}-\frac{3\sqrt{3}}{4}\mathrm{Ls}_3\Big(\frac{2\pi}{3}\Big)-\frac{\psi''(2)}{24}=-1.296\,463\cdots\ ,\\
Q_3 &= \frac{189}{64}+\frac{19A}{16}-\frac{A}{2}\ln 3-\frac{F_J}{192}+\frac{F_K}{16}-\frac{F_L}{16}-\frac{F_M}{24}\\
&\quad -\frac{\pi^3}{12\sqrt{3}}-\frac{7\zeta(3)}{48}-\frac{\sqrt{3}}{2}\mathrm{Ls}_3\Big(\frac{2\pi}{3}\Big)-\frac{5\psi''(2)}{192}=-0.423\,114\cdots\ .
\end{aligned}
\tag{13}
$$

III. NEXT-NEXT-NEXT-TO-LEADING LOGARITHM RESUMMATION OF THE EFFECTIVE POTENTIAL

In the usual loop expansion, the l-loop quantum correction to the effective potential for a single-component massive ϕ^4 theory has the following structure [8]:

$$
V^{(l)}(\phi,\lambda,x,y)=\lambda^{l+1}\phi^4\sum_{m=0}^{l-1}\sum_{n=0}^{l}a_{lmn}x^{m-2}y^n\ , \tag{14}
$$

where

$$
x\equiv\frac{1}{1+2m^2/(\lambda\phi^2)}\ ,\quad y\equiv\ln\frac{m_\phi^2}{\mu^2}\ . \tag{15}
$$

With this observation, we can rearrange the order of summation over indices $\{l,m,n\}$ appearing in the expansion of the full effective potential, $V=\sum_{l=0}^{\infty}\hbar^l V^{(l)}$, so as to give a leading-logarithm series expansion [8,9,13]:

$$
V=\sum_{l=0}^{\infty}\hbar^l V^{(\!(l)\!)}\ ,
$$

where the lth-to-leading logarithm contribution, $V^{(\!(l)\!)}$, is given as follows:

$$
V^{(\!(l)\!)}=\lambda\phi^4\sum_{n=l}^{\infty}\lambda^n\hbar^n y^{n-l}\sum_{m=0}^{n-1}a_{nm(n-l)}x^{m-2}\ .
$$

In the previous work [13], the second-to-leading logarithm term, $V^{(\!(2)\!)}$ was obtained. Now in the present paper, we calculate the third-to-leading, i.e., next-next-next-to-leading, logarithm term, $V^{(\!(3)\!)}$. In order to obtain a renormalization-group-improved effective potential which is exact up to Lth-to-leading logarithm order, we need $(L+1)$-loop RG functions together with the L-loop effective potential [9]. The various β and γ functions (β_λ, γ_m, γ_ϕ, and β_Λ) are known up to the five-loop order for a massive $O(N)$ ϕ^4 theory in four dimensions of spacetime [27–29]:

$$
\begin{aligned}
\beta_\lambda &= \frac{3\lambda^2\hbar}{(4\pi)^2} - \frac{17\lambda^3\hbar^2}{3(4\pi)^4} + \frac{\lambda^4\hbar^3}{(4\pi)^6}\Big(\frac{145}{8} + 12\zeta(3)\Big) + \frac{\lambda^5\hbar^4}{(4\pi)^8}\Big(-\frac{3499}{48} \\
&\quad -78\zeta(3) + 18\zeta(4) - 120\zeta(5)\Big) + \frac{\lambda^6\hbar^5}{(4\pi)^{10}}\Big(\frac{764621}{2304} + \frac{7965\zeta(3)}{16} \\
&\quad +45\zeta^2(3) - \frac{1189\zeta(4)}{8} + 987\zeta(5) - \frac{675\zeta(6)}{2} + 1323\zeta(7)\Big) \\
&\equiv \beta_1\lambda^2\hbar + \beta_2\lambda^3\hbar^2 + \beta_3\lambda^4\hbar^3 + \beta_4\lambda^5\hbar^4 + \beta_5\lambda^6\hbar^5 \ , \\
\gamma_m &= \frac{\lambda\hbar}{(4\pi)^2} - \frac{5\lambda^2\hbar^2}{6(4\pi)^4} + \frac{7\lambda^3\hbar^3}{2(4\pi)^6} - \frac{\lambda^4\hbar^4}{(4\pi)^8}\Big(\frac{477}{32} + \frac{3\zeta(3)}{2} + 3\zeta(4)\Big) \\
&\quad + \frac{\lambda^5\hbar^5}{(4\pi)^{10}}\Big(-\frac{3709}{2304} + \frac{3\zeta(3)}{16} - \frac{\zeta(4)}{2}\Big) \\
&\equiv \gamma_{m1}\lambda\hbar + \gamma_{m2}\lambda^2\hbar^2 + \gamma_{m3}\lambda^3\hbar^3 + \gamma_{m4}\lambda^4\hbar^4 + \gamma_{m5}\lambda^5\hbar^5 \ , \\
\gamma_\phi &= \frac{\lambda\hbar}{(4\pi)^2} \times 0 + \frac{\lambda^2\hbar^2}{12(4\pi)^4} - \frac{\lambda^3\hbar^3}{16(4\pi)^6} + \frac{65\lambda^4\hbar^4}{192(4\pi)^8} \\
&\quad + \frac{\lambda^5\hbar^5}{(4\pi)^{10}}\Big(\frac{158849}{2304} + \frac{1519\zeta(3)}{48} - 9\zeta^2(3) + \frac{65\zeta(4)}{4} + \zeta(5) + \frac{75\zeta(6)}{2}\Big) \\
&\equiv \gamma_1\lambda\hbar + \gamma_2\lambda^2\hbar^2 + \gamma_3\lambda^3\hbar^3 + \gamma_4\lambda^4\hbar^4 + \gamma_5\lambda^5\hbar^5 \ , \\
\beta_\Lambda &= \frac{m^4\hbar}{2(4\pi)^2} + \frac{m^4\lambda\hbar^2}{(4\pi)^4} \times 0 + \frac{m^4\lambda^2\hbar^3}{16(4\pi)^6} + \frac{m^4\lambda^3\hbar^4}{(4\pi)^8}\Big(\frac{\zeta(3)}{2} - \frac{25}{24}\Big) \\
&\quad + \frac{m^4\lambda^4\hbar^5}{(4\pi)^{10}}\Big(\frac{26171}{4608} - \frac{77\zeta(3)}{32} + \frac{21\zeta(4)}{16} - 2\zeta(5)\Big) \\
&\equiv m^4(\beta_{\Lambda1}\hbar + \beta_{\Lambda2}\lambda\hbar^2 + \beta_{\Lambda3}\lambda^2\hbar^3 + \beta_{\Lambda4}\lambda^3\hbar^4 + \beta_{\Lambda5}\lambda^4\hbar^5) \ .
\end{aligned}
\tag{16}
$$

A. Running parameters perturbed up to three-loop order

Since we assume the effective potential $V(= \sum_{l=0}^{\infty}\hbar^l V^{(l)} = \sum_{l=0}^{\infty}\hbar^l V^{((l))})$ is independent of the renormalization scale μ for the fixed values of the bare parameters, arbitrary changes of this scale μ can be compensated for by appropriate (finite) changes in the quantities (λ, m^2, ϕ, and Λ) that characterize the theory. This leads to the RG equation for the effective potential $V(\mu, \lambda, m^2, \phi, \Lambda)$

$$
\Big[\mu\frac{\partial}{\partial\mu} + \beta_\lambda\frac{\partial}{\partial\lambda} + \gamma_m m^2\frac{\partial}{\partial m^2} - \gamma_\phi\phi\frac{\partial}{\partial\phi} + \beta_\Lambda\frac{\partial}{\partial\Lambda}\Big]V(\mu, \lambda, m^2, \phi, \Lambda) = 0 \ . \tag{17}
$$

Applying the method of characteristics to Eq. (17), we can write the solution of Eq. (17), $V(\mu, \lambda, m^2, \phi, \Lambda)$ as follows:

$$
V(\mu, \lambda, m^2, \phi, \Lambda) = V(\bar{\mu}, \bar{\lambda}, \bar{m}^2, \bar{\phi}, \bar{\Lambda}) \ , \tag{18}
$$

where the barred quantities are running parameters which satisfy the following differential equations with respect to a running scale t:

$$\hbar\frac{d\bar{\mu}}{dt} = \bar{\mu} \ , \quad \hbar\frac{d\bar{\lambda}}{dt} = \beta_\lambda(\bar{\lambda}) \ , \quad \hbar\frac{d\bar{m}^2}{dt} = \gamma_m(\bar{\lambda})\bar{m}^2 \ ,$$
$$\hbar\frac{d\bar{\phi}}{dt} = -\gamma_\phi(\bar{\lambda})\bar{\phi} \ , \quad \hbar\frac{d\bar{\Lambda}}{dt} = \beta_\Lambda(\bar{\lambda}, \bar{m}^2) \ , \tag{19}$$

and at the boundary point, $t = 0$, their values are given as $\bar{\mu}(t = 0) = \mu$, $\bar{m}^2(t = 0) = m^2$, $\bar{\phi}(t = 0) = \phi$, and $\bar{\Lambda}(t = 0) = \Lambda$.

The $\bar{\mu}$ differential equation is very simple and its solution is given as

$$\bar{\mu}^2(t) = \mu^2 \exp(2t/\hbar) \ . \tag{20}$$

In order to solve $\bar{\lambda}$ differential equation, we try a perturbative solution by writing

$$\bar{\lambda} = \bar{\lambda}^{\langle 0\rangle} + \hbar\bar{\lambda}^{\langle 1\rangle} + \hbar^2\bar{\lambda}^{\langle 2\rangle} + \hbar^3\bar{\lambda}^{\langle 3\rangle} + O(\hbar^4) \ ,$$

with the boundary conditions $\bar{\lambda}^{\langle 0\rangle}(0) = \lambda$, $\bar{\lambda}^{\langle 1\rangle}(0) = \bar{\lambda}^{\langle 2\rangle}(0) = \bar{\lambda}^{\langle 3\rangle}(0) = 0$. Then, with β_λ in Eq. (16), the equation we want to solve splits into four first-order linear differential equations within the desired order:

$$\begin{aligned}
\frac{d\bar{\lambda}^{\langle 0\rangle}}{dt} &= \beta_1\bar{\lambda}^{\langle 0\rangle 2} \ , \\
\frac{d\bar{\lambda}^{\langle 1\rangle}}{dt} &= 2\beta_1\bar{\lambda}^{\langle 0\rangle}\bar{\lambda}^{\langle 1\rangle} + \beta_2\bar{\lambda}^{\langle 0\rangle 3} \ , \\
\frac{d\bar{\lambda}^{\langle 2\rangle}}{dt} &= 2\beta_1\bar{\lambda}^{\langle 0\rangle}\bar{\lambda}^{\langle 2\rangle} + \beta_1\bar{\lambda}^{\langle 1\rangle 2} + 3\beta_2\bar{\lambda}^{\langle 0\rangle 2}\bar{\lambda}^{\langle 1\rangle} + \beta_3\bar{\lambda}^{\langle 0\rangle 4} \ , \\
\frac{d\bar{\lambda}^{\langle 3\rangle}}{dt} &= 2\beta_1\bar{\lambda}^{\langle 0\rangle}\bar{\lambda}^{\langle 3\rangle} + 2\beta_1\bar{\lambda}^{\langle 1\rangle}\bar{\lambda}^{\langle 2\rangle} + 3\beta_2\bar{\lambda}^{\langle 0\rangle 2}\bar{\lambda}^{\langle 2\rangle} \\
&\quad +3\beta_2\bar{\lambda}^{\langle 0\rangle}\bar{\lambda}^{\langle 1\rangle 2} + 4\beta_3\bar{\lambda}^{\langle 0\rangle 3}\bar{\lambda}^{\langle 1\rangle} + \beta_4\bar{\lambda}^{\langle 0\rangle 5} \ .
\end{aligned}$$

Solutions to $\bar{\lambda}^{\langle 0\rangle}$, $\bar{\lambda}^{\langle 1\rangle}$, and $\bar{\lambda}^{\langle 2\rangle}$ differential equations have been obtained already in Ref. [13]. The $\bar{\lambda}^{\langle 3\rangle}$ differential equation is readily integrated as follows:

$$\begin{aligned}
\bar{\lambda}^{\langle 3\rangle} = \frac{\lambda^4}{(4\pi)^6}\bigg\{ &\frac{1}{T^2}\left[\frac{95807}{23328} + \frac{49\zeta(3)}{9} - 3\zeta(4) + 20\zeta(5)\right] \\
&+\frac{1}{T^3}\left[\frac{27251}{5832} + \frac{68\zeta(3)}{9} - \left(\frac{27251}{2916} + \frac{136\zeta(3)}{9}\right)\ln T\right] \\
&+\frac{1}{T^4}\bigg[-\frac{204811}{23328} - 13\zeta(3) + 3\zeta(4) - 20\zeta(5) + \left(\frac{121057}{5832} + \frac{68\zeta(3)}{3}\right)\ln T \\
&-\frac{24565}{1458}\ln^2 T + \frac{4913}{729}\ln^3 T\bigg]\bigg\} \ ,
\end{aligned} \tag{21}$$

where $T \equiv 1 - 3\lambda t/(4\pi)^2$. Similarly, we write $\bar{m}^2$ as

$$\bar{m}^2 = \bar{m}^{2\langle 0\rangle} + \hbar\bar{m}^{2\langle 1\rangle} + \hbar^2\bar{m}^{2\langle 2\rangle} + \hbar^3\bar{m}^{2\langle 3\rangle} + O(\hbar^4) \ ,$$

and with γ_m in Eq. (16), obtain splitted first-order linear differential equations:

$$
\begin{aligned}
\frac{d\bar{m}^{2\langle 0\rangle}}{dt} &= \gamma_{m1}\bar{\lambda}^{\langle 0\rangle}\bar{m}^{2\langle 0\rangle} \ , \\
\frac{d\bar{m}^{2\langle 1\rangle}}{dt} &= \gamma_{m1}\bar{\lambda}^{\langle 0\rangle}\bar{m}^{2\langle 1\rangle} + \gamma_{m1}\bar{\lambda}^{\langle 1\rangle}\bar{m}^{2\langle 0\rangle} + \gamma_{m2}\bar{\lambda}^{\langle 0\rangle 2}\bar{m}^{2\langle 0\rangle} \ , \\
\frac{d\bar{m}^{2\langle 2\rangle}}{dt} &= \gamma_{m1}\bar{\lambda}^{\langle 0\rangle}\bar{m}^{2\langle 2\rangle} + \gamma_{m1}\bar{\lambda}^{\langle 2\rangle}\bar{m}^{2\langle 0\rangle} + \gamma_{m1}\bar{\lambda}^{\langle 1\rangle}\bar{m}^{2\langle 1\rangle} + 2\gamma_{m2}\bar{\lambda}^{\langle 0\rangle}\bar{\lambda}^{\langle 1\rangle}\bar{m}^{2\langle 0\rangle} \\
&\quad +\gamma_{m2}\bar{\lambda}^{\langle 0\rangle 2}\bar{m}^{2\langle 1\rangle} + \gamma_{m3}\bar{\lambda}^{\langle 0\rangle 3}\bar{m}^{2\langle 0\rangle} \ , \\
\frac{d\bar{m}^{2\langle 3\rangle}}{dt} &= \gamma_{m1}\bar{\lambda}^{\langle 0\rangle}\bar{m}^{2\langle 3\rangle} + \gamma_{m1}\bar{\lambda}^{\langle 1\rangle}\bar{m}^{2\langle 2\rangle} + \gamma_{m1}\bar{\lambda}^{\langle 2\rangle}\bar{m}^{2\langle 1\rangle} + \gamma_{m1}\bar{\lambda}^{\langle 3\rangle}\bar{m}^{2\langle 0\rangle} \\
&\quad +\gamma_{m2}\bar{\lambda}^{\langle 0\rangle 2}\bar{m}^{2\langle 2\rangle} + 2\gamma_{m2}\bar{\lambda}^{\langle 0\rangle}\bar{\lambda}^{\langle 1\rangle}\bar{m}^{2\langle 1\rangle} + 2\gamma_{m2}\bar{\lambda}^{\langle 0\rangle}\bar{\lambda}^{\langle 2\rangle}\bar{m}^{2\langle 0\rangle} \\
&\quad +\gamma_{m2}\bar{\lambda}^{\langle 1\rangle 2}\bar{m}^{2\langle 0\rangle} + \gamma_{m3}\bar{\lambda}^{\langle 0\rangle 3}\bar{m}^{2\langle 1\rangle} + 3\gamma_{m3}\bar{\lambda}^{\langle 0\rangle 2}\bar{\lambda}^{\langle 1\rangle}\bar{m}^{2\langle 0\rangle} + \gamma_{m4}\bar{\lambda}^{\langle 0\rangle 4}\bar{m}^{2\langle 0\rangle} \ .
\end{aligned} \tag{22}
$$

With the $\bar{\lambda}$ solutions ($\bar{\lambda}^{\langle 0\rangle}$, $\bar{\lambda}^{\langle 1\rangle}$, $\bar{\lambda}^{\langle 2\rangle}$, and $\bar{\lambda}^{\langle 3\rangle}$), and the lower-order $\bar{m}^2$ solutions ($\bar{m}^{2\langle 0\rangle}$, $\bar{m}^{2\langle 1\rangle}$, and $\bar{m}^{2\langle 2\rangle}$, which have appeared also in Ref. [13]), and together with the boundary condition $\bar{m}^{2\langle 3\rangle}(0) = 0$, we obtain the following solution of $\bar{m}^{2\langle 3\rangle}$:

$$
\begin{aligned}
\bar{m}^{2\langle 3\rangle} &= \frac{\lambda^3 m^2}{(4\pi)^6}\Big\{\frac{1}{T^{1/3}}\Big[-\frac{245089}{944784} - \frac{89\zeta(3)}{54} + \zeta(4) - \frac{40\zeta(5)}{9}\Big] \\
&\quad +\frac{1}{T^{4/3}}\Big[\frac{539479}{314928} + \frac{68\zeta(3)}{27} - \zeta(4) + \frac{20\zeta(5)}{3} + \Big(\frac{30379}{314928} + \frac{34\zeta(3)}{81}\Big)\ln T\Big] \\
&\quad +\frac{1}{T^{7/3}}\Big[\frac{73843}{629856} + \frac{11\zeta(3)}{27} - \Big(\frac{38777}{19683} + \frac{272\zeta(3)}{81}\Big)\ln T - \frac{5491}{19683}\ln^2 T\Big] \\
&\quad +\frac{1}{T^{10/3}}\Big[-\frac{2968225}{1889568} - \frac{23\zeta(3)}{18} - \frac{20\zeta(5)}{9} + \Big(\frac{1284061}{314928} + \frac{238\zeta(3)}{81}\Big)\ln T \\
&\quad -\frac{85255}{39366}\ln^2 T + \frac{68782}{59049}\ln^3 T\Big]\Big\} \ .
\end{aligned} \tag{23}
$$

If we notice that the $\bar{\phi}$ differential equation and the $\bar{m}^2$ differential equation in Eq. (19) are of the same structure except the minus sign on the right-hand side, then we can readily write down linear differential equations for $\bar{\phi}^{\langle 0\rangle}$, $\bar{\phi}^{\langle 1\rangle}$, $\bar{\phi}^{\langle 2\rangle}$, and $\bar{\phi}^{\langle 3\rangle}$ in the perturbative decomposition

$$
\bar{\phi} = \bar{\phi}^{\langle 0\rangle} + \hbar\bar{\phi}^{\langle 1\rangle} + \hbar^2\bar{\phi}^{\langle 2\rangle} + \hbar^3\bar{\phi}^{\langle 3\rangle} + O(\hbar^4) \ .
$$

Lower-order solutions, $\bar{\phi}^{\langle 0\rangle}$, $\bar{\phi}^{\langle 1\rangle}$, and $\bar{\phi}^{\langle 2\rangle}$, are found in Ref. [13]. The $\bar{\phi}^{\langle 3\rangle}$ solution is obtained as follows:

$$
\begin{aligned}
\bar{\phi}^{\langle 3\rangle} &= \frac{\lambda^3\phi}{(4\pi)^6}\Big\{\frac{95}{46656} - \frac{\zeta(3)}{27} - \frac{7}{15552T} + \frac{1}{T^2}\Big[\frac{59}{864} + \frac{\zeta(3)}{9} - \frac{17}{11664}\ln T\Big] \\
&\quad +\frac{1}{T^3}\Big[-\frac{815}{11664} - \frac{2\zeta(3)}{27} + \frac{119}{2916}\ln T - \frac{289}{2916}\ln^2 T\Big]\Big\} \ ,
\end{aligned} \tag{24}
$$

which satisfy the boundary condition $\bar{\phi}^{\langle 3\rangle}(0) = 0$. Finally, we try the solution to the $\bar{\Lambda}$ differential equation as

$$
\bar{\Lambda} = \bar{\Lambda}^{\langle 0\rangle} + \hbar\bar{\Lambda}^{\langle 1\rangle} + \hbar^2\bar{\Lambda}^{\langle 2\rangle} + \hbar^3\bar{\Lambda}^{\langle 3\rangle} + O(\hbar^4) \ .
$$

The solutions to splitted lower-order equations

$$
\begin{aligned}
\frac{d\bar{\Lambda}^{\langle 0\rangle}}{dt} &= \beta_{\Lambda 1}\bar{m}^{2\langle 0\rangle 2}\ , \\
\frac{d\bar{\Lambda}^{\langle 1\rangle}}{dt} &= 2\beta_{\Lambda 1}\bar{m}^{2\langle 0\rangle}\bar{m}^{2\langle 1\rangle} + \beta_{\Lambda 2}\bar{\lambda}^{\langle 0\rangle}\bar{m}^{2\langle 0\rangle 2}\ , \\
\frac{d\bar{\Lambda}^{\langle 2\rangle}}{dt} &= \beta_{\Lambda 1}\bar{m}^{2\langle 1\rangle 2} + 2\beta_{\Lambda 1}\bar{m}^{2\langle 0\rangle}\bar{m}^{2\langle 2\rangle} + \beta_{\Lambda 2}\bar{\lambda}^{\langle 1\rangle}\bar{m}^{2\langle 0\rangle 2} \\
&\quad +2\beta_{\Lambda 2}\bar{\lambda}^{\langle 0\rangle}\bar{m}^{2\langle 0\rangle}\bar{m}^{2\langle 1\rangle} + \beta_{\Lambda 3}\bar{\lambda}^{\langle 0\rangle 2}\bar{m}^{2\langle 0\rangle 2}\ ,
\end{aligned}
$$

can be found in Ref. [13]. The solution to the $\bar{\Lambda}^{\langle 3\rangle}$ differential equation

$$
\begin{aligned}
\frac{d\bar{\Lambda}^{\langle 3\rangle}}{dt} &= 2\beta_{\Lambda 1}\bar{m}^{2\langle 0\rangle}\bar{m}^{2\langle 3\rangle} + 2\beta_{\Lambda 1}\bar{m}^{2\langle 1\rangle}\bar{m}^{2\langle 2\rangle} + \beta_{\Lambda 2}\bar{\lambda}^{\langle 2\rangle}\bar{m}^{2\langle 0\rangle 2} \\
&\quad +\beta_{\Lambda 2}\bar{\lambda}^{\langle 0\rangle}\bar{m}^{2\langle 1\rangle 2} + 2\beta_{\Lambda 2}\bar{\lambda}^{\langle 1\rangle}\bar{m}^{2\langle 0\rangle}\bar{m}^{2\langle 1\rangle} + 2\beta_{\Lambda 2}\bar{\lambda}^{\langle 0\rangle}\bar{m}^{2\langle 0\rangle}\bar{m}^{2\langle 2\rangle} \\
&\quad +2\beta_{\Lambda 3}\bar{\lambda}^{\langle 0\rangle}\bar{\lambda}^{\langle 1\rangle}\bar{m}^{2\langle 0\rangle 2} + 2\beta_{\Lambda 3}\bar{\lambda}^{\langle 0\rangle 2}\bar{m}^{2\langle 0\rangle}\bar{m}^{2\langle 1\rangle} + \beta_{\Lambda 4}\bar{\lambda}^{\langle 0\rangle 3}\bar{m}^{2\langle 0\rangle 2}
\end{aligned}
$$

with the boundary condition $\bar{\Lambda}^{\langle 3\rangle}(0) = 0$ is given as follows:

$$
\begin{aligned}
\bar{\Lambda}^{\langle 3\rangle} = \frac{\lambda^2 m^4}{(4\pi)^6}\Big\{ & -\frac{709}{720} - \frac{73\zeta(3)}{20} + \frac{3\zeta(4)}{2} - \frac{15\zeta(5)}{2} \\
& + T^{1/3}\Big[\frac{592037}{1889568} + \frac{305\zeta(3)}{162} - \zeta(4) + \frac{40\zeta(5)}{9}\Big] \\
& + \frac{1}{T^{2/3}}\Big[\frac{1579007}{1259712} + \frac{121\zeta(3)}{54} - \frac{\zeta(4)}{2} + \frac{10\zeta(5)}{3} + \Big(\frac{42653}{314928} + \frac{34\zeta(3)}{81}\Big)\ln T\Big] \\
& - \frac{1}{T^{5/3}}\Big[\frac{807937}{1574640} + \frac{76\zeta(3)}{135} + \Big(\frac{106301}{157464} + \frac{68\zeta(3)}{81}\Big)\ln T + \frac{5491}{39366}\ln^2 T\Big] \\
& + \frac{1}{T^{8/3}}\Big[-\frac{260647}{3779136} + \frac{29\zeta(3)}{324} - \frac{5\zeta(5)}{18} + \Big(\frac{91171}{157464} + \frac{34\zeta(3)}{81}\Big)\ln T \\
& - \frac{2023}{78732}\ln^2 T + \frac{24565}{118098}\ln^3 T\Big]\Big\}\ .
\end{aligned}
\tag{25}
$$

B. Summary of the next-next-next-to-leading logarithm resummation

If we choose t as

$$
t = \frac{\hbar}{2}\ln\Big(\frac{m_\phi^2}{\mu^2}\Big)\ , \tag{26}
$$

as was done in Ref. [13], we see that $\bar{\mu}^2(t)$ in Eq. (20) becomes

$$
\bar{\mu}^2(t) = m^2 + \frac{\lambda\phi^2}{2}\ , \tag{27}
$$

which is independent of μ. Eqs. (21), (23) – (25), and (27), together with the lower-loop results in Eq. (21) of Ref. [13], comprise the desired perturbative solutions to the evolution equations for running parameters. Now all things necessary for an improvement of the three-loop effective potential were obtained. We follows the same calculation procedure as in Ref. [13] for the correct collection of logarithms of various powers into a given leading-logarithm series order. The calculation is straightforward. The final result for $V^{((3))}(\phi,\lambda,m^2;t)$ in the leading-logarithm series expansion

$$\begin{aligned} V = & V^{((0))}(\phi,\lambda,m^2,\Lambda;t) + \hbar V^{((1))}(\phi,\lambda,m^2;t) + \hbar^2 V^{((2))}(\phi,\lambda,m^2;t) \\ & + \hbar^3 V^{((3))}(\phi,\lambda,m^2;t) + O(\hbar^4)\ , \end{aligned} \tag{28}$$

is summarized as follows:

$$\begin{aligned} V^{((3))} = & \frac{\lambda^3}{(4\pi)^6}\Big[\frac{m^4}{\lambda}\Big\{-\frac{709}{720} - \frac{73\zeta(3)}{20} + \frac{3\zeta(4)}{2} - \frac{15\zeta(5)}{2} \\ & + T^{1/3}\Big(\frac{592037}{1889568} + \frac{305\zeta(3)}{162} - \zeta(4) + \frac{40\zeta(5)}{9}\Big) + T^{-2/3}\Big(\frac{687889}{629856} + \frac{47\zeta(3)}{27} \\ & - \frac{\zeta(4)}{2} + \frac{10\zeta(5)}{3} + \Big[\frac{42653}{314928} + \frac{34\zeta(3)}{81}\Big]\ln T + \Big[\frac{2509}{23328} + \frac{\zeta(3)}{3}\Big]\ln S\Big) \\ & T^{-5/3}\Big(\frac{639571}{3149280} + \frac{59\zeta(3)}{135} - \Big[\frac{53975}{157464} + \frac{68\zeta(3)}{81}\Big]\ln T - \frac{5491}{39366}\ln^2 T \\ & - \Big[\frac{4201}{11664} + \frac{2\zeta(3)}{3}\Big]\ln S - \frac{323}{1458}\ln T\ln S - \frac{19}{216}\ln^2 S\Big) \\ & + T^{-8/3}\Big(Q_1 - \frac{1178747}{1889568} - \frac{133\zeta(3)}{324} - \frac{5\zeta(5)}{18} + \Big[\frac{162775}{157464} + \frac{34\zeta(3)}{81}\Big]\ln T \\ & - \frac{121091}{157464}\ln^2 T + \frac{24565}{118098}\ln^3 T + \Big[\frac{991}{1458} + \frac{\zeta(3)}{3}\Big]\ln S - \frac{1207}{1458}\ln T\ln S \\ & + \frac{1445}{2916}\ln^2 T\ln S - \frac{115}{432}\ln^2 S + \frac{85}{216}\ln T\ln^2 S + \frac{5}{48}\ln^3 S\Big)\Big\} \\ & + m^2\phi^2\Big\{T^{-1/3}\Big(-\frac{243379}{1889568} - \frac{91\zeta(3)}{108} + \frac{\zeta(4)}{2} - \frac{20\zeta(5)}{9}\Big) + T^{-4/3}\Big(\frac{484237}{629856} \\ & + \frac{103\zeta(3)}{108} - \frac{\zeta(4)}{2} + \frac{10\zeta(5)}{3} + \Big[\frac{16541}{314928} + \frac{17\zeta(3)}{81}\Big]\ln T + \Big[\frac{973}{23328} \\ & + \frac{\zeta(3)}{6}\Big]\ln S\Big) + T^{-7/3}\Big(\frac{96625}{78732} - \frac{2A}{27} + \frac{64\zeta(3)}{27} - \Big[\frac{54553}{78732} + \frac{136\zeta(3)}{81}\Big]\ln T \\ & - \frac{2312}{19683}\ln^2 T - \Big[\frac{3641}{5832} + \frac{4\zeta(3)}{3}\Big]\ln S - \frac{136}{729}\ln T\ln S - \frac{2}{27}\ln^2 S\Big) \\ & + T^{-10/3}\Big(Q_2 - \frac{882083}{472392} + \frac{2A}{27} - \frac{67\zeta(3)}{27} - \frac{10\zeta(5)}{9} \\ & + \Big[\frac{262225}{39366} + \frac{119A}{108} + \frac{119\zeta(3)}{81}\Big]\ln T - \frac{65314}{19683}\ln^2 T + \frac{34391}{59049}\ln^3 T \\ & + \Big[\frac{3242}{729} + \frac{7A}{8} + \frac{7\zeta(3)}{81}\Big]\ln S - \frac{12155}{2916}\ln T\ln S + \frac{2023}{1458}\ln^2 T\ln S \\ & - \frac{317}{216}\ln^2 S + \frac{119}{108}\ln T\ln^2 S + \frac{7}{24}\ln^3 S\Big)\Big\} + \lambda\phi^4\Big\{T^{-1}\Big(\frac{53}{93312} - \frac{\zeta(3)}{162}\Big) \\ & + T^{-2}\Big(\frac{42785}{279936} + \frac{5\zeta(3)}{24} - \frac{\zeta(4)}{8} + \frac{5\zeta(5)}{6} + \frac{85}{15552}\ln T + \frac{5}{1152}\ln S\Big) \\ & + T^{-3}\Big(\frac{204905}{279936} + \frac{A}{72} + \frac{121\zeta(3)}{108} - \Big[\frac{31297}{69984} + \frac{17\zeta(3)}{27}\Big]\ln T \end{aligned}$$

$$
\begin{aligned}
&+\frac{289}{17496}\ln^2 T - \left[\frac{1787}{5184} + \frac{\zeta(3)}{2}\right]\ln S + \frac{17}{648}\ln T\ln S + \frac{1}{96}\ln^2 S\Big)\\
&+T^{-4}\Big(Q_3 - \frac{247849}{279936} - \frac{A}{72} - \frac{857\zeta(3)}{648} + \frac{\zeta(4)}{8} - \frac{5\zeta(5)}{6}\\
&+\left[\frac{249169}{69984} + \frac{17A}{24} + \frac{17\zeta(3)}{18}\right]\ln T - \frac{122825}{69984}\ln^2 T + \frac{4913}{17496}\ln^3 T\\
&+\left[\frac{2807}{1296} + \frac{9A}{16} + \frac{3\zeta(3)}{4}\right]\ln S - \frac{731}{324}\ln T\ln S + \frac{289}{432}\ln^2 T\ln S - \frac{145}{192}\ln^2 S\\
&+\frac{17}{32}\ln T\ln^2 S + \frac{9}{64}\ln^3 S\Big)\Big\} + \Big\{\frac{m^4}{\lambda}\Big[-\frac{19}{108}T^{-2/3} + T^{-5/3}\Big(-\frac{2}{27} + \frac{17}{54}\ln T\\
&+\frac{1}{4}\ln S\Big)\Big] + m^2\phi^2\Big[-\frac{2}{27}T^{-4/3} + T^{-7/3}\Big(-\frac{73}{108} + \frac{17}{27}\ln T + \frac{1}{2}\ln S\Big)\Big]\\
&+\lambda\phi^4\Big[\frac{1}{144}T^{-2} + T^{-3}\Big(-\frac{23}{72} + \frac{17}{72}\ln T + \frac{3}{16}\ln S\Big)\Big]\Big\}\frac{Z}{W}\\
&+\Big\{\frac{m^4}{\lambda}T^{-2/3} + m^2\phi^2 T^{-4/3} + \frac{\lambda\phi^4}{4}T^{-2}\Big\}\Big(\frac{P}{4W} - \frac{Z^2}{8W^2}\Big)\Big]\ ,
\end{aligned}
\tag{29}
$$

where

$$
\begin{aligned}
T &\equiv 1 - \frac{3\lambda t}{(4\pi)^2}\ , \quad W \equiv m^2 T^{-1/3} + \frac{\lambda\phi^2}{2}T^{-1}\ ,\\
Z &\equiv m^2\Big[-\frac{19}{54}T^{-1/3} + T^{-4/3}\Big(\frac{19}{54} + \frac{17}{27}\ln T\Big)\Big] + \lambda\phi^2\Big[\frac{1}{36}T^{-1} + T^{-2}\Big(-\frac{1}{36} + \frac{17}{18}\Big)\Big]\ ,\\
P &\equiv m^2\Big[T^{-1/3}\Big(\frac{1787}{11664} + \frac{2\zeta(3)}{3}\Big) - T^{-4/3}\Big(\frac{5531}{5832} + \frac{4\zeta(3)}{3} + \frac{323}{1458}\ln T\Big)\\
&\quad +T^{-7/3}\Big(\frac{9275}{11664} + \frac{2\zeta(3)}{3} - \frac{221}{729}\ln T + \frac{578}{729}\ln^2 T\Big)\Big] + \lambda\phi^2\Big[\frac{43}{2592}T^{-1}\\
&\quad +T^{-2}\Big(-\frac{535}{432} - 2\zeta(3) + \frac{17}{324}\ln T\Big) + T^{-3}\Big(\frac{3167}{2592} + 2\zeta(3) - \frac{17}{9}\ln T + \frac{289}{162}\ln^2 T\Big)\Big]\ ,\\
S &\equiv \frac{W}{m^2 + \lambda\phi^2/2}\ .
\end{aligned}
$$

The leading, next-to-leading, and next-next-to-leading terms [$V^{((0))}$, $V^{((1))}$, and $V^{((2))}$] in Eq. (28) are given in Eq. (25) of Ref. [13].

IV. CONCLUDING REMARKS

We have applied the RG method for a systematic resummation of the perturbation expansion. The next-next-next-to leading logarithm resummation, $V^{((3))}$, given in Eq. (29) is our main result. If we expand $V^{((3))}$ in powers of t and put this t back to the value on the right-hand side of Eq. (26), then we obtain an infinite series of logarithm terms at the scale μ, i.e., at $t = 0$, each of which is the next-next-next-to-leading logarithm term in the corresponding loop expansion $V^{(i)}$. Our result reproduce correctly the known terms of lowest three orders in the series:

$$
\begin{aligned}
V^{((3))} = & \frac{\lambda^3}{(4\pi)^6}\Big[Q_1\frac{m^4}{\lambda} + Q_2 m^2\phi^2 + Q_3\lambda\phi^4\Big] \\
& + \frac{\lambda^4\hbar}{(4\pi)^8}\Big[\Big\{C_1\frac{m^4}{\lambda} + C_2 m^2\phi^2 + C_3\lambda\phi^4 + C_4\frac{\lambda\phi^4}{1+2m^2/(\lambda\phi^2)}\Big\}\ln\Big(\frac{m_\phi^2}{\mu^2}\Big)\Big] \\
& + \frac{\lambda^5\hbar^2}{(4\pi)^{10}}\Big[\Big\{c_1\frac{m^4}{\lambda} + c_2 m^2\phi^2 + c_3\lambda\phi^4 + c_4\frac{\lambda\phi^4}{1+2m^2/(\lambda\phi^2)} \\
& + c_5\frac{\lambda\phi^4}{[1+2m^2/(\lambda\phi^2)]^2}\Big\}\ln^2\Big(\frac{m_\phi^2}{\mu^2}\Big)\Big] + \cdots ,
\end{aligned}
$$

where the values of C_1, ..., C_5 and c_1, ..., c_5 are, respectively, given in Eq. (32) and Eq. (34) below.

Purely numerical computation of finite parts for some three-loop vacuum diagrams in a paper by Pelissetto and Vicari [30] enables us to extract the numerical values for F_J, F_K, F_L, and F_M. The extracted results are

$$
\begin{aligned}
F_J &= \frac{251}{12} + 9A - 20\zeta(3) + \psi''(2) + 80S_1 + 20S_2 - 120S_5 - 30S_6 - 6S_7 , \\
F_K &= -\frac{226}{3} - 14A + 8\zeta(3) - \frac{\psi''(2)}{2} - 24S_1 - 6S_2 + 6S_4 - 12S_7 , \\
F_L &= -1 + 6A - \frac{4\zeta(3)}{3} + \frac{\psi''(2)}{6} + 8S_1 + 2S_2 + 2S_4 , \\
F_M &= 6\zeta(3) + 8H ,
\end{aligned} \tag{30}
$$

where the quantities S_1, S_2, S_4, S_5, S_6, S_7, and H are known numerically in the Appendix B of Ref. [30].[6] In obtaining F_J and F_K, we assume first the *unknown* finite parts F_J and F_K for J and K in Eq. (10) since the pole parts of them are already known [4,6]. Then we differentiate these J and K with respect to m_ϕ^2. Meanwhile, we can differentiate J and K in Eq. (2) with respect to m_ϕ^2 before momentum integrations, yielding three-loop diagrams given in Appendix B of Ref. [30] whose finite parts are known numerically. See Figs. 1 and 2. By equating the results of differentiations thus done, the unknowm values, F_J and F_K, are determined. We see that numerical values, F_J, F_K, F_L and F_M given in Eq. (30), agree with the analytical values in Eq. (11).

Another perturbative use of the (nonperturbative) RG equation in Eq. (17) without introducing running scale t enables us to determine a considerable number of coefficients in the following general form of the four-loop and five-loop corrections to the effective potential [see Eqs. (14) and (15)]:

$$
V^{(4)} = \frac{\lambda^4}{(4\pi)^8}\Big[A_1\frac{m^4}{\lambda} + A_2 m^2\phi^2 + A_3\lambda\phi^4 + A_4\frac{\lambda\phi^4}{1+2m^2/(\lambda\phi^2)}
$$

---

[6]The value of $H = -0.9825$ in Eq. (B.23) of Ref. [30] is incorrect. The correct value is $H = -2.1559$. The authors of Ref. [30] informed us that the error is due to a misprint in their writing the final expression H: they used $H = -0.130388 - \frac{\pi^4}{80} + \frac{3}{4}\zeta(3)\ln\frac{3}{2}$ instead of $H = -1.30388 - \frac{\pi^4}{80} + \frac{3}{4}\zeta(3)\ln\frac{3}{2}$.

$$
\begin{aligned}
&+\left\{B_1\frac{m^4}{\lambda}+B_2m^2\phi^2+B_3\lambda\phi^4+B_4\frac{\lambda\phi^4}{1+2m^2/(\lambda\phi^2)}\right\}\ln\left(\frac{m_\phi^2}{\mu^2}\right)\\
&+\left\{C_1\frac{m^4}{\lambda}+C_2m^2\phi^2+C_3\lambda\phi^4+C_4\frac{\lambda\phi^4}{1+2m^2/(\lambda\phi^2)}\right\}\ln^2\left(\frac{m_\phi^2}{\mu^2}\right)\\
&+\left\{D_1\frac{m^4}{\lambda}+D_2m^2\phi^2+D_3\lambda\phi^4+D_4\frac{\lambda\phi^4}{1+2m^2/(\lambda\phi^2)}\right\}\ln^3\left(\frac{m_\phi^2}{\mu^2}\right)\\
&+\left\{E_1\frac{m^4}{\lambda}+E_2m^2\phi^2+E_3\lambda\phi^4+E_4\frac{\lambda\phi^4}{1+2m^2/(\lambda\phi^2)}\right\}\ln^4\left(\frac{m_\phi^2}{\mu^2}\right)\Bigg]\,,\\
V^{(5)}=\frac{\lambda^5}{(4\pi)^{10}}&\Bigg[a_1\frac{m^4}{\lambda}+a_2m^2\phi^2+a_3\lambda\phi^4+a_4\frac{\lambda\phi^4}{1+2m^2/(\lambda\phi^2)}+a_5\frac{\lambda\phi^4}{[1+2m^2/(\lambda\phi^2)]^2}\\
&+\left\{b_1\frac{m^4}{\lambda}+b_2m^2\phi^2+b_3\lambda\phi^4+b_4\frac{\lambda\phi^4}{1+2m^2/(\lambda\phi^2)}+b_5\frac{\lambda\phi^4}{[1+2m^2/(\lambda\phi^2)]^2}\right\}\ln\left(\frac{m_\phi^2}{\mu^2}\right)\\
&+\left\{c_1\frac{m^4}{\lambda}+c_2m^2\phi^2+c_3\lambda\phi^4+c_4\frac{\lambda\phi^4}{1+2m^2/(\lambda\phi^2)}+c_5\frac{\lambda\phi^4}{[1+2m^2/(\lambda\phi^2)]^2}\right\}\ln^2\left(\frac{m_\phi^2}{\mu^2}\right)\\
&+\left\{d_1\frac{m^4}{\lambda}+d_2m^2\phi^2+d_3\lambda\phi^4+d_4\frac{\lambda\phi^4}{1+2m^2/(\lambda\phi^2)}+d_5\frac{\lambda\phi^4}{[1+2m^2/(\lambda\phi^2)]^2}\right\}\ln^3\left(\frac{m_\phi^2}{\mu^2}\right)\\
&+\left\{e_1\frac{m^4}{\lambda}+e_2m^2\phi^2+e_3\lambda\phi^4+e_4\frac{\lambda\phi^4}{1+2m^2/(\lambda\phi^2)}+e_5\frac{\lambda\phi^4}{[1+2m^2/(\lambda\phi^2)]^2}\right\}\ln^4\left(\frac{m_\phi^2}{\mu^2}\right)\\
&+\left\{f_1\frac{m^4}{\lambda}+f_2m^2\phi^2+f_3\lambda\phi^4+f_4\frac{\lambda\phi^4}{1+2m^2/(\lambda\phi^2)}+f_5\frac{\lambda\phi^4}{[1+2m^2/(\lambda\phi^2)]^2}\right\}\ln^5\left(\frac{m_\phi^2}{\mu^2}\right)\Bigg]\,.
\end{aligned}
\tag{31}
$$

Since all coefficients in $V^{(4)}$ except A_1 to A_4 (these require computing the non-logarithmic portion of the four-loop diagrams) were already determined from the four-loop order part of the RG equation (see Eq. (27) of Ref. [13]):

$$
\begin{aligned}
&B_1=-\frac{295}{192}+4Q_1+\frac{\zeta(3)}{4}\,,\\
&B_2=-\frac{271}{32}-\frac{53A}{48}+5Q_2-\frac{15\zeta(3)}{8}-\frac{3\zeta(4)}{4}\,,\\
&B_3=-\frac{1549}{768}-\frac{35A}{96}+6Q_3-\frac{9\zeta(3)}{4}+\frac{3\zeta(4)}{8}-\frac{5\zeta(5)}{2}\,,\\
&B_4=\frac{A}{8}+\frac{\zeta(3)}{4}\,,\\
&C_1=\frac{73}{48}\,,\qquad C_2=\frac{97}{8}+\frac{35A}{16}+\frac{3\zeta(3)}{4}\,,\\
&C_3=\frac{583}{96}+\frac{27A}{16}+\frac{9\zeta(3)}{8}\,,\qquad C_4=-\frac{1}{16}\,,\\
&D_1=-\frac{55}{96}\,,\qquad D_2=-\frac{277}{96}\,,\qquad D_3=-\frac{201}{128}\,,\qquad D_4=\frac{1}{48}\,,\\
&E_1=\frac{5}{48}\,,\qquad E_2=\frac{35}{96}\,,\qquad E_3=\frac{27}{128}\,,\qquad E_4=0\,.
\end{aligned}
\tag{32}
$$

we here determine the coefficients of $V^{(5)}$ using the RG equation. The five-loop order part of the complete RG equation, Eq. (17), is given by[7]

$$\begin{aligned}
\mu\frac{\partial V^{(5)}}{\partial\mu} &+\lambda\Big\{\beta_1\frac{\partial V^{(4)}}{\partial\lambda}+\gamma_{m1}m^2\frac{\partial V^{(4)}}{\partial m^2}-\gamma_1\phi\frac{\partial V^{(4)}}{\partial\phi}+\beta_{\Lambda 1}\frac{\partial V^{(4)}}{\partial\Lambda}\Big\} \\
&+\lambda^2\Big\{\beta_2\frac{\partial V^{(3)}}{\partial\lambda}+\gamma_{m2}m^2\frac{\partial V^{(3)}}{\partial m^2}-\gamma_2\phi\frac{\partial V^{(3)}}{\partial\phi}+\beta_{\Lambda 2}\frac{\partial V^{(3)}}{\partial\Lambda}\Big\} \\
&+\lambda^3\Big\{\beta_3\frac{\partial V^{(2)}}{\partial\lambda}+\gamma_{m3}m^2\frac{\partial V^{(2)}}{\partial m^2}-\gamma_3\phi\frac{\partial V^{(2)}}{\partial\phi}+\beta_{\Lambda 3}\frac{\partial V^{(2)}}{\partial\Lambda}\Big\} \\
&+\lambda^4\Big\{\beta_4\frac{\partial V^{(1)}}{\partial\lambda}+\gamma_{m4}m^2\frac{\partial V^{(1)}}{\partial m^2}-\gamma_4\phi\frac{\partial V^{(1)}}{\partial\phi}+\beta_{\Lambda 4}\frac{\partial V^{(1)}}{\partial\Lambda}\Big\} \\
&+\lambda^5\Big\{\beta_5\frac{\partial V^{(0)}}{\partial\lambda}+\gamma_{m5}m^2\frac{\partial V^{(0)}}{\partial m^2}-\gamma_5\phi\frac{\partial V^{(0)}}{\partial\phi}+\beta_{\Lambda 5}\frac{\partial V^{(0)}}{\partial\Lambda}\Big\}=0\ . \qquad (33)
\end{aligned}$$

Substituting $V^{(0)} - V^{(3)}$ in Eq. (12), $V^{(4)}$ [with the determined coefficients in Eq. (32)] and $V^{(5)}$ in Eq. (31) into Eq. (33), we readily obtain all coefficients[8] except a_1 to a_5, (for the latter one must determine the non-logarithmic portion of the five-loop diagrams):

$$\begin{aligned}
b_1 &= \frac{6915}{1024}+\frac{11A_1}{2}-\frac{9Q_1}{2}+\frac{3\zeta(3)}{64}+\frac{45\zeta(4)}{32}-\zeta(5)\ , \\
b_2 &= -\frac{168127}{9216}+\frac{781A}{192}+\frac{13A_2}{2}+2Q_1-\frac{13Q_2}{2}+\frac{307\zeta(3)}{192} \\
&\quad +3A\zeta(3)+\frac{9\zeta^2(3)}{2}-\frac{21\zeta(4)}{2}+\frac{29\zeta(5)}{2}-\frac{75\zeta(6)}{4}\ , \\
b_3 &= \frac{78179}{36864}+\frac{173A}{128}+\frac{15A_3}{2}-Q_1+\frac{5Q_2}{2}-\frac{17Q_3}{2}+\frac{33851\zeta(3)}{2304} \\
&\quad +\frac{9A\zeta(3)}{4}+\frac{27\zeta^2(3)}{16}-\frac{2213\zeta(4)}{384}+\frac{1283\zeta(5)}{48}-\frac{325\zeta(6)}{32}+\frac{441\zeta(7)}{16}\ , \\
b_4 &= \frac{11}{6}-\frac{A}{16}+\frac{17A_4}{2}+Q_1-\frac{5Q_2}{2}+6Q_3-\frac{7\zeta(3)}{4}+\frac{3\zeta(4)}{4}-\frac{5\zeta(5)}{2}\ , \\
b_5 &= \frac{A}{8}-A_4+\frac{\zeta(3)}{4}\ , \\
c_1 &= -\frac{2281}{288}+11Q_1-\frac{\zeta(3)}{4}-\frac{3\zeta(4)}{8}\ , \\
c_2 &= -\frac{113159}{2304}-\frac{1235A}{192}+\frac{65Q_2}{4}-\frac{237\zeta(3)}{16}-\frac{3\zeta(4)}{2}-\frac{15\zeta(5)}{2}\ , \\
c_3 &= -\frac{60073}{4608}-\frac{341A}{128}+\frac{45Q_3}{2}-\frac{109\zeta(3)}{8}+\frac{63\zeta(4)}{32}-\frac{105\zeta(5)}{8}\ ,
\end{aligned}$$

[7]The values of β_i and $\beta_{\Lambda i}$ $(i = 1, ..., 5)$ in Eq. (33) have additional overall multiplying factors than those values in Eq. (16). This is because we use a different convention for β_i and $\beta_{\Lambda i}$ in Eq. (33):

$$\beta_\lambda = \beta_1\lambda\hbar+\beta_2\lambda^2\hbar^2+\cdots\ ,\ \text{ and }\ \beta_\Lambda = \beta_{\Lambda 1}\lambda\hbar+\beta_{\Lambda 2}\lambda^2\hbar^2+\cdots\ .$$

[8]In fact, the values of b_1 to b_5 in Eq. (34) are not completely fixed ones since the coefficients of non-logarithm part of $V^{(4)}$, i.e., A_1 to A_4 are unknown yet.

$$
\begin{aligned}
&c_4 = \frac{33}{64} + \frac{9A}{8} + \frac{29\zeta(3)}{16}\,, \qquad c_5 = -\frac{1}{16} - \frac{A}{16} - \frac{\zeta(3)}{8}\,, \\
&d_1 = \frac{4319}{1152} + \frac{\zeta(3)}{4}\,, \qquad d_2 = \frac{4403}{144} + \frac{455A}{96} + \frac{21\zeta(3)}{8}\,, \\
&d_3 = \frac{75793}{4608} + \frac{135A}{32} + \frac{27\zeta(3)}{8}\,, \qquad d_4 = -\frac{47}{96}\,, \qquad d_5 = \frac{1}{24}\,, \\
&e_1 = -\frac{695}{768}\,, \qquad e_2 = -\frac{3925}{768}\,, \qquad e_3 = -\frac{3161}{1024}\,, \qquad e_4 = \frac{19}{192}\,, \qquad e_5 = -\frac{1}{192}\,, \\
&f_1 = \frac{11}{96}\,, \qquad f_2 = \frac{91}{192}\,, \qquad f_3 = \frac{81}{256}\,, \qquad f_4 = 0\,, \qquad f_5 = 0\,. \qquad (34)
\end{aligned}
$$

Indeed, the leading logarithm coefficients (f_1 to f_5), the next-to-leading logarithm coefficients (e_1 to e_5), the next-next-to-leading logarithm coefficients (d_1 to d_5), and the next-next-next-to-leading logarithm coefficients (c_1 to c_5) can be confirmed from the expansion of $V^{((0))}$, $V^{((1))}$, $V^{((2))}$, and $V^{((3))}$, in powers of t, respectively.

As remarked earlier, with L-loop effective potential and $(L+1)$-loop RG functions, we can obtain an RG improved effective potential which is exact up to Lth-to-leading logarithm order [9]. Let us recall that all the coefficients of $V^{(4)}$ except A_1 to A_4 in Eq. (7) of Ref. [13] are known in Eq. (27) of Ref. [13]. Thus, if these four unknown coefficients are evaluated by any means, we can obtain $V^{((4))}$ since the five-loop RG functions are known in Eq. (16).

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FIGURES

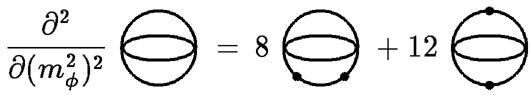


FIG. 1. Differentiation of J with respect to m_ϕ^2

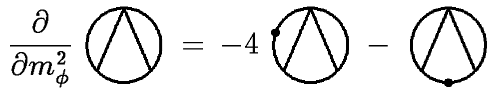
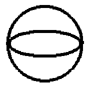
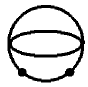
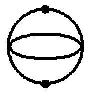
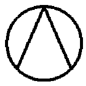
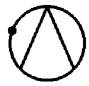


FIG. 2. Differentiation of K with respect to m_ϕ^2

$\dfrac{\partial^2}{\partial(m_\phi^2)^2}$  $= 8$  $+ 12$ 

$\frac{\partial}{\partial m_\phi^2}$  $= -4$  $-$